# ON THE MOTION OF A VORTEX BELOW THE SURFACE OF A LIQUID

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The problem of steady motion of a vortex below the surface of a liquid in the nonlinear formulation was considered by Moiseev [1 ].

Moiseev [1 ] showed that for Froude numbers slightly larger than unity two solutions are possible in every case. One of the solutions describes a flow passing into plane-parallel flow as the intensity of the vortex tends to zero; the other solution describes flow which tends under the same conditions into a solitary wave. A theorem of existence and uniqueness of the first solution "in the small" was established by Ter-Krikorov [2 ]. The present author [3 ] established a theorem of existence and uniqueness for the second solution, but also only "in the small", i.e. for small values of the intensity of the vortex.

Below, we establish a theorem of the existence of the first solution for finite values of the intensity of the vortex. The proof of the theorem is based on an application of the topological methods of Leray and Schauder's fixed-point theory [4 ].

**1. Formulation of the problem.** The problem of steady motion of a vortex below the surface of a heavy ideal liquid in dimensionless variables reduces to the determination of the analytic function $\zeta(z) = \xi(x, y) + i\eta(x, y)$, which conformally transforms the physical $z$-plane (Fig. 1) on a strip of unit width $0 < \eta < 1$ in the parametric $\zeta$-plane (Fig. 2). Setting

$$\frac{d\zeta}{dz} = e^{-i\omega(\zeta)}, \qquad \omega(\zeta) = \theta(\xi, \eta) + i\tau(\xi, \eta)$$

we reduce the problem to the determination of the analytic function $\omega(\zeta)$ which satisfies the following conditions (see [2 ]):

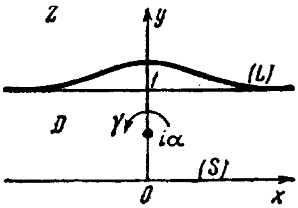


Fig. 1.

$$\frac{\partial\theta}{\partial\eta}=\nu\,\frac{e^{-3\tau}\sin\theta}{f^2(\xi)}+\frac{f'(\xi)}{f(\xi)},\qquad \nu<1 \text{ when } \eta=1 \tag{1.1}$$

$$\omega=0 \text{ when } |\xi|\to\infty,\quad \theta=0 \text{ when } \eta=0 \tag{1.2}$$

Here

$$f(\xi)=1-\frac{\gamma}{2}\,\frac{\sin\pi\beta}{\cosh\,\pi\xi+\cos\pi\beta},\qquad \gamma=\frac{\Gamma}{cH} \tag{1.3}$$

$$\beta=\alpha+\operatorname{Im}\int_{i\beta}^{i-\infty}[e^{i\omega(t)}-1]\,dt$$

$\alpha$ and $\beta$ are the depths of submergence of the vortex in the physical and the parametric planes, respectively.

In order that the velocity at the free surface $L$ should not vanish, it is necessary to fulfil the condition $\gamma < 2\cot\pi/2\beta$. This condition is always fulfilled when $\gamma < 0$. This case is of special interest, since the lift force is directed upwards.

The function $\omega(\zeta)$ determines, up to a constant, the single-valued function $z(\zeta)$. A sufficient condition to ensure the one-sheeted nature of the function $z(\zeta)$ is the condition [5 ]

$$|\theta(\xi,1)|\leqslant\pi \tag{1.4}$$

We notice that in the given dimensionless formulation the solution of the problem is determined by the parameters $\nu$, $V_0 = e^{-\tau(0)}$, $\gamma$ and $\beta$.

**2. Green's function. The general equations of the problem.** The fundamental boundary condition (1.1) can be written in the form

$$\frac{\partial\theta}{\partial\eta}=\nu\,\frac{e^{-3\tau}\sin\theta}{f^2(\xi)}+\frac{f'(\xi)}{f(\xi)}=F(\xi) \text{ when } \eta=1,\qquad \theta=0 \text{ when } \eta=0 \tag{2.1}$$

We shall define the Green's function of the problem (2.1), as the function $G(\zeta,\zeta') = H(\zeta,\zeta') + iQ(\zeta,\zeta')$, analytic in the strip $0 < \eta < 1$, having a logarithmic singularity at the point $\xi = \xi'$, $\eta = \eta'$, and satisfying the conditions

$$\frac{\partial H}{\partial\eta}=0 \text{ when } \eta=1,\qquad H=0 \text{ when } \eta=0 \tag{2.2}$$

The Green's function $G$ satisfying conditions (2.2) has the form

$$G(\zeta,\zeta')=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\sin\mu(\zeta-\xi')}{\mu}\,\frac{\cosh\mu(1-\eta')}{\cosh\mu}\,d\mu \tag{2.3}$$

Setting $\eta' = 1$ in this, we obtain the theorem.

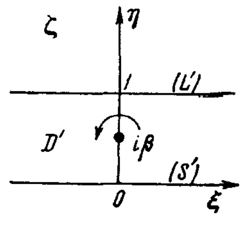


**Fig. 2.**

*Theorem 2.1.* If $F(\xi)$ is an absolutely integrable function, the the function

$$\omega(\zeta) = \int_{-\infty}^{+\infty} G(\zeta, \xi' + i) F(\xi') d\xi'$$

is analytic in the open strip $0 < \eta < 1$, continuous in the closed strip $0 \leqslant \eta \leqslant 1$, and if $F(\xi)$ is an odd function, then the real part of $\omega(\zeta)$ satisfies conditions (1.2) and (2.1).

This theorem is proved in a manner similar to the theorems (3.1) and (3.2) in [2].

Separating the imaginary and real parts of (2.3), we obtain

$$\theta = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_0(\xi - \xi', \eta) F(\xi') d\xi', \quad \tau = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} Q_0(\xi - \xi', \eta) F(\xi') d\xi' \tag{2.4}$$

where

$$H_0(\xi - \xi', \eta) = \int_{-\infty}^{+\infty} \frac{\cos\mu(\xi - \xi') \sinh \mu\eta}{\mu \cosh \mu} d\mu = \ln \frac{\cosh {}^1/_2\pi(\xi - \xi') + \sin {}^1/_2\pi\eta}{\cosh {}^1/_2\pi(\xi - \xi') - \sin {}^1/_2\pi\eta}$$

$$Q_0(\xi - \xi', \eta) = \int_{-\infty}^{+\infty} \frac{\sin\mu(\xi - \xi') \cosh \mu\eta}{\mu \cosh \mu} d\mu = 2\tan^{-1}\left\{\frac{\sinh {}^1/_2\pi(\xi - \xi')}{\cos {}^1/_2\pi\eta}\right\}$$

The first integral (2.4), according to Theorem (2.1), is continuous when $\eta = 1$, and the second when $\eta = 0$. Accordingly, using the notation $\theta(\xi, 1) = \theta(\xi)$, $\tau(\xi, 1) = \tau(\xi)$, $\tau(\xi, 0) = \tau_0(\xi)$, we obtain

$$\theta(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \ln \frac{\cosh {}^1/_2\pi(\xi - \xi') + 1}{\cosh {}^1/_2\pi(\xi - \xi') - 1} F(\xi') d\xi' \tag{2.5}$$

$$\tau_0(\xi) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \tan^{-1}\{\sinh {}^1/_2\pi(\xi - \xi')\} F(\xi') d\xi' \tag{2.6}$$

Moreover, making use of the relation $\partial\theta/\partial\eta = -\partial\tau/\partial\xi$, from (1.1) we find the relation between $\theta(\xi)$ and $\tau(\xi)$:

$$-\tau(\xi) = \ln f(\xi) + \nu \int_{-\infty}^{\xi} \frac{e^{-3\tau} \sin\theta}{f^2(\xi')} d\xi' \tag{2.7}$$

Let us notice certain properties of the required function in the half-strip $0 \leqslant \eta \leqslant 1$, $-\infty < \xi \leqslant 0$. From condition (1.2) it follows that the integral

$$\int_{-\infty}^{0} \sin\theta(\xi)\, d\xi$$

is finite and that the function $r(\xi)$ is continuous and equal to zero when $|\xi| = \infty$. From (2.7), also, it follows that the function $r(\xi)$ is continuous and differentiable with respect to $\xi$ and bounded:

$$|\tau(\xi)| \leqslant \rho, \qquad \rho = -\ln V_0 \tag{2.8}$$

Then from (2.6) the function $r_0(\xi)$ is also bounded and, as is easily seen, $|r_0(\xi)| \leqslant \rho$.

The function $\theta(\xi)$ is bounded on $L'$ and the following inequality holds:

$$|\theta(\xi)| \leqslant |F(\xi)| \leqslant \nu / V_0^3 + p(\gamma) \tag{2.9}$$

The inequality (2.9) is easily obtained from (2.5); moreover

$$p(\gamma) = \frac{\pi|\gamma| \sin {}^1\!/_2\pi\beta}{2\cos {}^1\!/_2\pi\beta\,(2\cos^2 {}^1\!/_2\pi\beta - {}^1\!/_2\gamma\sin\pi\beta)}, \qquad 0 < V_0 < 1$$

The condition of one-sheetedness (1.4) of the function $z(\zeta)$ imposes a supplementary restriction on the initial parameters $\nu$, $V_0$, $\gamma$ of the form

$$\nu / V_0^3 + p(\gamma) \leqslant \pi \tag{2.10}$$

*Note*. By virtue of (2.9) the following inequality also holds:

$$|\theta(\xi)| \leqslant \nu / V_0^3 \max|\theta(\xi)| + p(\gamma)$$

Hence, it follows that when $\nu < 1$

$$\max|\theta(\xi)| \to 0 \text{ when } \gamma \to 0, \text{ if } \nu / V_0^3 < 1 \tag{2.11}$$

i.e. when $\gamma = 0$ the flow becomes plane-parallel.

On the other hand, from (2.10) it follows that as $\gamma$ increases in absolute value $\nu$ must decrease, i.e. the solution satisfying condition (2.10) corresponds to high-speed flows.

Accordingly, if the function $F(\xi)$ is odd, i.e. the wave is symmetrical, the functions $\theta(\xi)$, $r(\xi)$, $r_0(\xi)$ satisfy Equations (2.7), (2.5) and (2.6) and all the boundary and asymptotic conditions. Let us denote by $V$ the system of equations (2.5), (2.6), (2.7) where $a$ is the functional (1.3), whilst the initial parameters satisfy the condition (2.10).

We shall study the question of the existence of a solution of the system $V$, assuming that $\theta(\xi)$ belongs to the class of functions having a given *a priori* modulus of continuity (or majorant) at infinity of the form $|\theta(\xi)| < CT(\xi)$, where $C$ is, as yet, an arbitrary constant and $T(\xi)$ satisfies the following conditions:

a) the function $T(\xi)$ is continuous when $-\infty < \xi < +\infty$, positive and even, decreasing when $\xi > 0$, increasing when $\xi < 0$, and $T(\pm\infty) = 0$;

b) the integral $\int_{-\infty}^{0} T(\xi)\, d\xi$ is finite.

Introducing the modulus of continuity of $\theta(\xi)$, the system $V$ can be replaced by a system $V_c$, depending on $C$ and $T(\xi)$, to which we apply Leray and Schauder's fixed-point theory [4]. We define the majorant $T(\xi)$ and $C$ so that a solution of the system $V_c$ is a solution of the system $V$, and we prove that the system $V_c$ has a solution in every case.

**3. Construction of the system $V_c$. Functional equations of the problem.** Let us assume that there exists a constant $C$ – a positive number – and a continuous function $T(\xi)$, satisfying conditions (2.11), such that

$$|\theta(\xi)| < CT(\xi) \tag{3.1}$$

Let us consider a certain solution of the system $V$ which satisfies condition (3.1). Let $\theta(\xi)$ satisfy condition (1.4), and $\tau(\xi)$ and $\theta(\xi)$ be connected by relation (2.7). Then condition (2.7), according to (3.1), can be written in the form

$$-\tau(\xi) = \ln f(\xi) + \nu \int_{-\infty}^{\xi} \sup\left\{\frac{e^{-3\tau}\sin\theta}{f^2(\xi')},\ CT(\xi')\right\} d\xi' \tag{3.2}$$

where $-\tau(\xi)$ is a positive function, and $\sup\{Q_1(\xi),\ Q_2(\xi)\}$ denotes the lower envelope of the two positive functions $Q_1(\xi)$ and $Q_2(\xi)$.

Let us denote by $V_c$ the system (2.5), (2.6) and (3.2). The system $V_c$ depends on $C$ and $T(\xi)$. It is not difficult to see that

$$\left|\frac{f'(\xi)}{f(\xi)}\right| = \frac{\pi|\gamma|\sin\pi\beta}{2} e^{-\pi|\xi|}\varphi(\xi), \qquad 0 \leqslant \varphi(\xi) \leqslant 1, \qquad \varphi(0) = 0$$

Then, according to (2.1), it is natural to set

$$T(\xi) = e^{-\pi|\xi|} \tag{3.3}$$

Moreover, the function $T(\xi)$ is chosen so as to satisfy all the conditions (2.11).

Let us assume that we have a certain solution of the system $V_c$, and that the initial parameters $\nu$, $V_0$ and $\gamma$ satisfy condition (2.9). This solution differs from a solution of the system $V$ only in that on the upper boundary of the region $D'$ (Fig. 2) we have condition (3.2); hence it follows that

$$\left|\frac{\partial\tau}{\partial\xi}\right| \leqslant CT(\xi)\left\{\frac{\nu}{V_0^2}+\frac{\mu}{C}\right\}, \qquad \mu=\frac{\pi|\gamma|\sin\pi\beta}{2} \tag{3.4}$$

The inequality (3.4) enables us to estimate the magnitude of $\theta(\xi)$ from (2.5). Since $F(\xi)=\partial\tau/\partial\xi$ it is easy to show that

$$|\theta(\xi)| \leqslant CT(\xi)\{\nu/V_0^3+\mu/C\} \tag{3.5}$$

From inequality (3.5) it follows that when $\nu/V_0^3<1$ the constant $C$ can be chosen in such a way as to fulfil condition (3.1).

Accordingly, under condition (2.10) there exists a system $V_c$ which possesses all the necessary properties. Let us prove that the system $V_c$ can be reduced to a functional equation.

For this purpose let us introduce the space $B$ of functions $\phi(\xi)$ which are continuous and finite in the interval $-\infty<\xi\leqslant 0$, and with a norm $\|\phi(\xi)\| = \max|\phi(\xi)|$; the space $B$ is linear, normalized and complete.

We introduce the notation

$$-\tau(\xi)=H_1(\xi), \qquad \theta(\xi)=H_2(\xi), \qquad -\tau_0(\xi)=H_3(\xi)$$

Then the system $V_c$ can be replaced by the following system of equations:

$$H_1(\xi)=\ln f(\xi)+\nu\int_{-\infty}^{\xi}\sup\left\{\frac{e^{-3\tau}\sin\theta}{f^2(\xi')}\,,\ CT(\xi')\right\}d\xi' \tag{3.6}$$

$$H_2(\xi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln\frac{\cosh{}^1/_2\pi(\xi-\xi')+1}{\cosh{}^1/_2\pi(\xi-\xi')-1}F(\xi')\,d\xi' \tag{3.7}$$

$$H_3(\xi)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\tan^{-1}\{\sinh{}^1/_2\pi(\xi-\xi')\}F(\xi')\,d\xi' \tag{3.8}$$

$$F(\xi)=\nu\frac{e^{-3\tau}\sin\theta}{f^2(\xi)}+\frac{f'(\xi)}{f(\xi)} \tag{3.9}$$

Let us consider a certain solution of the problem. From the form of (3.7) and (3.8) it follows that $H_2(\xi)$ and $H_3(\xi)$ belong to $B$. From (3.6) it also follows that the differentiable function $H_1\in B$.

If we denote by $R = B \times B \times B$ the space of vectors with components $r(\xi)$, $\theta(\xi)$, $r_0(\xi)$ and with norm $\|y\|$ where $y \in R$, then the system (3.6) to (3.9) can be written in the form of a functional equation

$$y = W(y), \qquad y \in R \tag{3.10}$$

which places the element $y$ with coordinates $r(\xi)$, $\theta(\xi)$, $r_0(\xi)$ in correspondence with the element $y'$ with coordinates $H_1(\xi)$, $H_2(\xi)$, $H_3(\xi)$. The space $R$ is linear, normalized and complete.

**4. The existence of a solution.** We shall prove that the operator $W$ is completely continuous, i.e. it is continuous and compact on any bounded set of $R$.

By assumption the operator $W$ acts on a bounded set of $R$. Therefore, from the form of Equations (3.6) to (3.9), it is not difficult to establish that the functions $H_1(\xi)$, $H_2(\xi)$, $H_3(\xi)$ belong to the space $B$, and depend continuously upon $r(\xi)$, $\theta(\xi)$, and, moreover, $dH_1/d\xi$ is finite in the interval $-\infty < \xi \leqslant 0$. Let us prove that the operator $W$ is compact.

The function $H_1(\xi)$, defined by the right-hand side of (3.6), for any point $y \in R$ satisfies the relations

$$|dH_1/d\xi| \leqslant \nu/V_0^3 + p(\gamma, \beta) \leqslant \pi \tag{4.1}$$

where

$$p = \frac{\pi\,|\gamma|\sin {}^1/_2\pi\beta}{2\cos {}^1/_2\pi\beta\,(2\cos^2 {}^1/_2\pi\beta - {}^1/_2\gamma\sin\pi\beta)}$$

$$\left|\frac{dH_2}{d\xi}\right| \leqslant CT(\xi)\left(\frac{\nu}{V_0^3} + \frac{\mu}{C}\right) \leqslant CT(\xi) \tag{4.2}$$

under the condition that $\nu/V_0^3 < 1$ and a $C$ such that condition (3.1) is fulfilled.

From inequalities (4.1) and (4.2) and the properties of the function $T(\xi)$, it follows that $H_1(\xi) \in R$ and the operator $H_1$ is continuous to the same degree and uniformly bounded on any finite sphere of $R$.

Similarly it can be shown that $H_2(\xi)$ and $H_3(\xi)$ belong to $R$, are continuous to the same degree, and are uniformly bounded on any sphere of $R$.

Hence, on the basis of Artsel's theorem, it follows that the operator $W$ is compact, and since $W$ is also continuous then consequently it is also completely continuous on every sphere of $R$.

*Theorem 4.1.* There exists at least one solution of the problem, the initial parameters of which satisfy the inequalities

$$\nu / V_0^3 < 1, \qquad \nu / V_0^3 + p(\gamma, \beta) \leqslant \pi \tag{4.3}$$

Let us consider a certain solution of the functional equation (3.10), the initial parameters of which satisfy the inequality

$$\nu / V_0^3 + p(\gamma, \beta) \leqslant \pi \tag{4.4}$$

When the inequality (4.4) is fulfilled the given solution defines a certain flow pattern of the liquid, on the boundary of which the condition (3.2) or (3.6) is fulfilled. Moreover, since from (3.7) we have the inequality

$$| d\tau / d\xi | \leqslant \nu / V_0^3 + p(\gamma, \beta) = A(\nu, V_0, \gamma, \beta)$$

we can prove that $\|y\| \leqslant$ const $A(\nu, V_0, \gamma, \beta)$ + const.

Let us further consider the equation

$$y = W(\mu', y) \tag{4.5}$$

which is obtained by replacing $\nu$ and $\gamma$ by $\mu'\nu$ and $\mu'\gamma$. It is not difficult to show that when the inequality (4.4) is fulfilled, which with the substitution of $\nu$ and $\gamma$ by $\mu'\nu$ and $\mu'\gamma$ is an entire increasing function of $\mu'$, the solutions of Equation (4.5) are uniformly bounded when $0 \leqslant \mu' \leqslant 1$, the operator $W$ is completely continuous for every $\mu'$ in the interval $0 \leqslant \mu' \leqslant 1$, and uniformly continuous with respect to $\mu'$ for every value of $y$ in a sphere of the space $R$.

Setting $\mu' = 0$ in Equation (4.5), we obtain $y = 0$. Consequently [ 4 ], the complete index of the functional equation (4.5) when $\mu' = 0$ is equal to $+1$, and Equation (3.10), obtained from (4.5) when $\mu' = 1$, has at least one solution. Accordingly, the problem is solved.

*Note (4.1).* Conditions (4.3) and (4.4) imply that $\nu < 1$ and that as $\gamma$ increases in magnitude $\nu$ diminishes. Such solutions correspond to high-speed-flow patterns.

*Note (4.2).* From the condition $0 \leqslant |\theta(\xi)| \leqslant \pi$ on the half-strip $0 \leqslant \eta \leqslant 1$, $-\infty < \xi \leqslant 0$, it follows that the form of the given half-strip does not overlap. But it can happen that the given half-strip (see Fig. 2) does overlap the symmetric half-strip. A sufficient condition for the non-overlapping of the half-strips $0 \leqslant \eta \leqslant 1$, $-\infty < \xi \leqslant 0$ and $0 \leqslant \eta \leqslant 1$, $0 \leqslant \xi < +\infty$ is the condition $|\theta(\xi)| < 1/2\,\pi$, and consequently condition (2.9) may be replaced by the following:

$$\nu / V_0^3 + p(\gamma, \beta) \leqslant {}^1/_2\,\pi$$

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